QP CODE: 26019083

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Reg No : .....

Name : .....

M.Sc DEGREE (CSS) EXAMINATION, MARCH 2026

Fourth Semester

M Sc MATHEMATICS

Core Courses - ME010401 - SPECTRAL THEORY

2019 Admission Onwards

70CCE71C

Time: 3 Hours

Weightage: 30

Part A (Short Answer Questions)

*Answer any **eight** questions.*

*Weight **1** each.*

1. When we can say that a subset of a metric space is rare, meager and nonmeager. State Baire's Category Theorem.
2. Define closed linear operator. Give an example of a closed linear operator which is not bounded.
3. Define the eigenvalues and eigenvectors of an $n \times n$ matrix.
4. Define resolvent operator and spectrum of a linear operator $T : D(T) \to X$, where $X \neq \{0\}$ is a complex normed space and $D(T) \subset X$.
5. Let matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $a, b, c, d.$ are real numbers. Find the eigenvalues of AB.
6. Let$A$ be an algebra with identity. Let $x \in A$ be such that x is invertible and commutes with y on A. Then show that x^{-1} and y also commutes.
7. Prove that every relatively compact subset of a metric space is totally bounded.
8. Establish the relation between the spectrum and point spectrum of a compact linear operator on a Banach space.
9. Prove that the relation "$\leq$" defined by $T_1 \leq T_2 \iff \langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle \ \forall x \in H$ is a partial ordering on the set of all bounded self-adjoint linear operators on a complex Hilbert space H.
10. For any projection P on a Hilbert space H, prove that $P \geq 0$ and $\|P\| \leq 1$.

(8×1=8 weightage)

Turn Over

Part B (Short Essay/Problems)

*Answer any **six** questions.*

*Weight **2** each.*

11. Prove that if the dual space X' of a normed space X is separable, then X itself is separable.

12. Prove that a sequence (f_n) of bounded linear functionals on a Banch space X is weak* convergent if and only if
    (A) The sequence $(\|f_n\|)$ is bounded.
    (B) The sequence $(f_n x)$ is cauchy for every x in a total subset M of X.

13. Define contraction on a metric space. Prove that contraction is a continuous mapping.

14. Prove that the eigenvectors $x_1, x_2, \cdots, x_n$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a linear operator T defined on a vector space, are linearly independant.

15. Let A be a complex Banach algebra with identity and $x \in A$. Then show that the set spectrum $\sigma(x)$ is compact and the spectral radius satisfies $r_\sigma(x) \leq \|x\|$.

16. Show that $T : l^2 \to l^2$ defined by $Tx = y = (\eta_j), \eta_j = \frac{\xi_j}{2}$ is compact, where $x = (\xi_j), j = 1, 2, 3, \cdots$.

17. Prove that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H is real.

18. Let $T : H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space $H \neq \{0\}$. Prove that m and M are spectral values of T where $m = \inf_{\|x\|=1} \langle Tx, x \rangle$ and $M = \sup_{\|x\|=1} \langle Tx, x \rangle$.

(6×2=12 weightage)

Part C (Essay Type Questions)

*Answer any **two** questions.*

*Weight **5** each.*

19. State and prove Bounded Inverse Theorem.

20. Let $T \in B(X, X)$, where X is a complex Banach space. Then:
    (a) Prove that the spectrum of T is closed.
    (b) Prove that the spectrum of T is compact.

21. If T is a is a compact linear operator on a normed space X, Prove that for every $\lambda \neq 0$,
    a) $\dim \mathcal{N}(T_\lambda^n) < \infty$ and $\{0\} = \mathcal{N}(T_\lambda^0) \subset \mathcal{N}(T_\lambda^0) \subset \mathcal{N}(T_\lambda^2) \subset \cdots$
    b) the range of T_λ^n is closed and $X = T_\lambda^0(X) \supset T_\lambda(X) \supset T_\lambda^2(X) \supset \cdots$.

22. Let P_1 and P_2 be projections defined on a Hilbert space H and let $Y_1 = P_1(H)$ and $Y_2 = P_2(H)$.

1. Prove that $P_2P_1 = P_1P_2 = P_1$ if and only if $Y_1 \subset Y_2$
2. Prove that the difference $P = P_1 - P_2$ is a projection if and only if $Y_1 \subset Y_2$

(2×5=10 weightage)